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J. Math. Anal. Appl. 294 (2004) 237–245
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Weakly G-KKM mappings, G-KKM property, and minimax inequalities

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Received 1 December 2003
Available online 8 April 2004
Submitted by D. O'Regan

1. Introduction

For a nonempty set X, 2^X denotes the class of all nonempty subsets of X and $\langle X\rangle$ denotes the class of all nonempty finite subsets of X.

If X is a subset of a vector space E, a set-valued mapping $S : X \to 2^E$ satisfying $\operatorname{co} A \subset S(A)$ for any $A \in \langle X\rangle$, is called a KKM mapping. It is well known that if S is a KKM mapping from a convex subset X of a topological space E into 2^E, then the family $\{\overline{S}(x)\colon x \in X\}$ has the finite intersection property (where $\overline{S}(x)$ denotes the closure of $S(x)$). Motivated by this result, Park [14] introduced the concept of generalized KKM mapping obtaining thus generalized KKM theorems and generalized matching theorems. Recently, Chang and Yen [5] made a systematic study of the class $\mathrm{KKM}(X, Y)$ which is defined as follows.

Let X be a convex subset of a vector space and Y a topological space. If $S, T : X \to 2^Y$ are two mappings such that $T(\operatorname{co} A) \subset S(A)$ for any $A \in \langle X\rangle$, then S is called a *generalized KKM mapping* w.r.t. T. A mapping $T : X \to 2^Y$ is said to have the *KKM property* if for any generalized KKM mapping w.r.t. T, $S : X \to 2^Y$ the family $\{\overline{S}(x)\colon x \in X\}$ has the finite intersection property.

In [13], Lin, Ko and Park extended the results of Chang and Yen to generalized convex spaces, introducing the concepts of generalized G-KKM mapping (w.r.t. T) and G-KKM property. In the next section we recall these concepts and introduce a new one, namely that of weakly G-KKM mapping (w.r.t. T). Relating to this, in Section 3, we obtain some intersection results and minimax inequalities of Ky Fan type. In Section 4 we give a new class of mappings with G-KKM property and establish a new Sion type minimax inequality.

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doi:10.1016/j.jmaa.2004.02.013

2. Preliminaries

If X and Y are topological spaces, a mapping $T : X \to 2^Y$ is said to be:

(i) *upper semicontinuous* (u.s.c.) if the set $\{x \in X\colon\ T(x) \cap F \neq \emptyset\}$ is closed in X, for each closed subset F of Y;
(ii) *lower semicontinuous* (l.s.c.) if the set $\{x \in X\colon\ T(x) \cap V \neq \emptyset\}$ is open in X, for each open subset V of Y;
(iii) *compact* if the image $T(X)$ of X under T is contained in a compact subset of Y.

A *generalized convex space* or a *G-convex space* (see [16,17]) $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$, there exists a subset $\Gamma(A)$ of X and a continuous function $\Phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\Phi_A(\Delta_J) \subset \Gamma(J)$. Here Δ_n denotes any n-simplex with vertices $\{e_i\}_{i=0}^n$ and Δ_J the face of Δ_n corresponding to $J \in \langle A\rangle$; that is, if $A = \{z_0, z_1, \ldots, z_n\}$ and $J = \{z_{i_0}, z_{i_1}, \ldots, z_{i_k}\} \subset A$ then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. In the case $D = X$, then $(X, D; \Gamma)$ will be denoted by $(X; \Gamma)$.

The main example of G-convex space corresponds to the case when $X = D$ is a convex subset of a Hausdorff topological vector space and for each $A \in \langle X\rangle$, $\Gamma(A)$ is the convex hull of A. For other major examples of G-convex spaces see [18,19].

Let $(X, D; \Gamma)$ be a G-convex space with $D \subset X$. A subset C of X is said to be *Γ-convex* (or *G-convex*) if for each $A \in \langle D\rangle$, $A \subset C$ implies $\Gamma(A) \subset C$. If Y is a nonempty set and $\beta \in \mathbb{R}$, we say that a function $\varphi : X \times Y \to \mathbb{R}$ is *G-β-quasiconvex* on X if for each $\lambda < \beta$ and $y \in Y$ the set $\{x \in X\colon\ \varphi(x, y) < \lambda\}$ is Γ-convex.

Definition 1. Let $(X, D; \Gamma)$ be a G-convex space, Y a nonempty set and $T : X \to 2^Y$, $S : D \to 2^Y$ two mappings. We say that S is a *generalized G-KKM mapping* w.r.t. T if for each $A \in \langle D\rangle$, $T(\Gamma(A)) \subset S(A)$. If Y is a topological space, $T : X \to 2^Y$ is said to have the *G-KKM property* if for any mapping $S : D \to 2^Y$ generalized G-KKM w.r.t. T, the family $\{\bar{S}(z)\colon\ z \in D\}$ has the finite intersection property.

Note that the notions introduced by Definition 1 coincide with the corresponding notions in [13, Definition 1] only when $D = X$.

We give now a new concept:

Definition 2. Let $(X, D; \Gamma)$ be a G-convex space, Y a nonempty set and $T : X \to 2^Y$, $S : D \to 2^Y$ two mappings. We say that S is *weakly G-KKM mapping* w.r.t. T if for each $A \in \langle D\rangle$ and any $x \in \Gamma(A)$, $T(x) \cap S(A) \neq \emptyset$.

Clearly each generalized G-KKM mapping w.r.t. T is weakly G-KKM mapping w.r.t. T.

3. Intersection results and Fan type minimax inequalities

The following extension to G-convex spaces of Fan's matching theorem is well known. For instance, it is equivalent with assertion (i) of Theorem 1 in [20].

Lemma 1. *Let $(X, D; \Gamma)$ be a G-convex space, $A \in \langle D \rangle$ and $\{M_z\colon z \in A\}$ an open or closed cover of X. Then there exists a nonempty subset B of A such that $\Gamma(B) \cap \bigcap\{M_z\colon z \in B\} \neq \emptyset$.*

Theorem 2. *Let $(X, D; \Gamma)$ be a compact G-convex space, Y a nonempty set and $T : X \to 2^Y$, $S : D \to 2^Y$ two mappings satisfying the following conditions*:

(i) *S is weakly G-KKM mappings w.r.t. T;*
(ii) *for each $z \in D$ the set $\{x \in X\colon T(x) \cap S(z) \neq \emptyset\}$ is closed.*

Then there exists an $x_0 \in X$ such that $T(x_0) \cap S(z) \neq \emptyset$ for each $z \in D$.

Proof. Suppose the conclusion does not hold and for every $z \in D$ put

$$M_z = \bigl\{x \in X\colon T(x) \cap S(z) = \emptyset\bigr\}.$$

Then the family $\{M_z\colon z \in D\}$ is an open cover of X and since X is compact there is a set $A \in \langle D \rangle$ such that $\bigcup\{M_z\colon z \in A\} = X$. By Lemma 1 there exists a nonempty subset B of A and a point

$$x_0 \in \Gamma(B) \cap \bigcap\{M_z\colon z \in B\}.$$

Since S is weakly G-KKM mapping w.r.t. T, by $x_0 \in \Gamma(B)$ we get $T(x_0) \cap S(B) \neq \emptyset$. On the other hand, since $x_0 \in \bigcap\{M_z\colon z \in B\}$, we have $T(x_0) \cap S(z) = \emptyset$ for each $z \in B$, hence $T(x_0) \cap S(B) = \emptyset$. The obtained contradiction completes the proof. $\square$

Remark 1. Condition (ii) in Theorem 2 is fulfilled if Y is topological space, T is upper semicontinuous and S has closed values.

The next result can be considered as a version of Theorem 2.

Theorem 3. *Let $(X, D; \Gamma)$ be a G-convex space, Y a nonempty set and $T : X \to 2^Y$, $S : D \to 2^Y$ two mappings satisfying the following conditions*:

(i) *S is weakly G-KKM mappings w.r.t. T;*
(ii) *the sets $\{x \in X\colon T(x) \cap S(z) \neq \emptyset\}$ are either all closed or all open, for all $z \in D$.*

Then for each $A \in \langle D \rangle$ there exists a point $x_0 \in \Gamma(A)$ such that $T(x_0) \cap S(z) \neq \emptyset$ for all $z \in A$.

Proof. Consider $A \in \langle D \rangle$ and for each $B \in \langle A \rangle$ put $\widetilde{\Gamma}(B) = \Gamma(A) \cap \Gamma(B)$. It is easily seen that $(\Gamma(A), A; \widetilde{\Gamma})$ is a G-convex space. Now the conclusion can be obtained using the same argument as in the proof of Theorem 2. $\square$

Remark 2. Condition (ii) in Theorem 3 is fulfilled if Y is a topological space and either T is u.s.c. and S is closed-valued or T is l.s.c. and S is open-valued.

In the next theorem as in the other minimax theorems established in our paper we shall suppose $\inf_x \sup_y \varphi(x, y) > -\infty$ but all these results remain (trivially) true when $\inf_x \sup_y \varphi(x, y) = -\infty$.

Theorem 4. *Let $(X, D; \Gamma)$ be a compact G-convex space and Y be a topological space. Let $T : X \to 2^Y$ be an u.s.c. mapping, $\psi : D \times Y \to \mathbb{R}$, $\varphi : X \times Y \to \mathbb{R}$ be two functions and $\beta = \inf_{x\in X} \sup_{y\in T(x)} \varphi(x, y)$. Suppose that*:

(1) *for each $z \in D$, $\psi(z, \cdot)$ is u.s.c. on Y;*
(2) *for any $\lambda < \beta$, $y \in T(x)$ and each $A \in \langle\{z \in D\colon\ \psi(z, y) < \lambda\}\rangle$ one has $\Gamma(A) \subset \{x \in X\colon\ \varphi(x, y) < \lambda\}$.*

Then the following statements hold:

(a) $\inf_{x\in X} \sup_{y\in T(x)} \varphi(x, y) \leqslant \sup_{x\in X} \inf_{z\in D} \sup_{y\in T(x)} \psi(z, y)$.
(b) *If $T(x)$ is compact for all $x \in X$, then there exists an $x_0 \in X$ such that*

$$\inf_{x\in X} \sup_{y\in T(x)} \varphi(x, y) \leqslant \inf_{z\in D} \sup_{y\in T(x_0)} \psi(z, y).$$

Proof. Let $\lambda < \beta$ be fixed and define $S : D \to 2^Y$ by

$$S(z) = \bigl\{y \in Y\colon\ \psi(z, y) \geqslant \lambda\bigr\}, \quad z \in D.$$

By (1), $S(z)$ is closed for each $z \in D$. We show that S is weakly G-KKM w.r.t. T. Suppose, on contrary, that there exist $A \in \langle D\rangle$ and $\bar{x} \in \Gamma(A)$ such that $T(\bar{x}) \cap S(A) = \emptyset$. Then for each $y \in T(\bar{x})$, $A \subset \{z \in D\colon\ \psi(z, y) < \lambda\}$. Consequently, by (2),

$$\bar{x} \in \Gamma(A) \subset \bigl\{x \in X\colon\ \varphi(x, y) < \lambda\bigr\} \quad \text{for all } y \in T(\bar{x}).$$

Hence $\sup_{y\in T(\bar{x})} \varphi(\bar{x}, y) \leqslant \lambda$, which contradicts $\lambda < \beta$.

By Theorem 2, via Remark 1, there exists a point $x_0 \in X$ such that $T(x_0) \cap S(z) \neq \emptyset$ for all $z \in D$. Hence, we have that $\lambda \leqslant \inf_{z\in D} \sup_{y\in T(x_0)} \psi(z, y)$, and thereby $\lambda \leqslant \sup_{x\in X} \inf_{z\in D} \sup_{y\in T(x)} \psi(z, y)$, which proves part (a).

Further, if $T(x)$ is compact for all $x \in X$, then $x \to \inf_{z\in D} \sup_{y\in T(x)} \psi(z, y)$ is u.s.c. on X because T is u.s.c. on X and $\psi(z, \cdot)$ is u.s.c. on Y (see [2, Proposition 3.1.21]). Since X is compact there exists an $x_0 \in X$ such that

$$\inf_{z\in D} \sup_{y\in T(x_0)} \psi(z, y) = \sup_{x\in X} \inf_{z\in D} \sup_{y\in T(x)} \psi(z, y).$$

Therefore, part (b) follows from part (a). $\square$

Corollary 5. *If $D \subset X$, the conclusion of Theorem* 4 *holds if condition* (2) *is replaced by the following two conditions*:

(3) $\varphi(z, y) \leqslant \psi(z, y)$ *for all* $(z, y) \in D \times T(X)$;
(4) $\varphi|_{X \times T(X)}$ *is* G-β*-quasiconvex on* X.

Proof. It suffices to prove that (3) and (4) imply (2). If $\lambda < \beta$, $y \in T(X)$ and $A \in \langle\{z \in D\colon \psi(z, y) < \lambda\}\rangle$, then, by (3), $A \in \langle\{z \in D\colon \varphi(z, y) < \lambda\}\rangle$, and by (4), $\Gamma(A) \subset \{x \in X\colon \varphi(x, y) < \lambda\}$. □

The origin of Theorem 4 and Corollary 5 goes back to Fan's minimax inequality [6]. Our results include earlier Fan type minimax inequalities due to Tan [21], Ha [7], Park [15], Liu [12], Kim [11].

The proofs of the following two variants of Theorem 4 are similar to that of Theorem 4 using as argument Theorem 3 instead of Theorem 2. For illustration we shall give the proof of Theorem 7.

Theorem 6. *Let* $(X, D; \Gamma)$ *be a* G*-convex space and* $Y, T, \psi, \varphi, \beta$ *be as in Theorem* 4*. Then*

$$\inf_{x \in X} \sup_{y \in T(x)} \varphi(x, y) \leqslant \inf_{A \in \langle D\rangle} \sup_{x \in \Gamma(A)} \min_{z \in A} \sup_{y \in T(x)} \psi(z, y).$$

If $D \subset X$*, then the conclusion holds if condition* (2) *is replaced by conditions* (3) *and* (4).

Theorem 7. *Let* $(X, D; \Gamma)$ *be a* G*-convex space and* Y *be a topological space. Let* $T : X \to 2^Y$ *be an l.s.c. mapping,* $\psi : D \times Y \to \mathbb{R}$*,* $\varphi : X \times Y \to \mathbb{R}$ *be two functions and* $\beta = \inf_{x \in X} \sup_{y \in T(x)} \varphi(x, y)$*. Suppose that*:

(5) *for each* $z \in D$*,* $\psi(z, \cdot)$ *is l.s.c. on* Y;
(6) *for any* $\lambda < \beta$*,* $y \in T(X)$ *and each* $A \in \langle\{z \in D\colon \psi(z, y) \leqslant \lambda\}\rangle$ *one has* $\Gamma(A) \subset \{x \in X\colon \varphi(x, y) \leqslant \lambda\}$.

Then

$$\inf_{x \in X} \sup_{y \in T(x)} \varphi(x, y) \leqslant \inf_{A \in \langle D\rangle} \sup_{x \in \Gamma(A)} \min_{z \in A} \sup_{y \in T(x)} \psi(z, y).$$

If $D \subset X$*, then the conclusion holds if condition* (6) *is replaced by conditions* (3) *and* (4).

Proof. Let $\lambda < \beta$ be fixed and define $S : D \to 2^Y$ by

$$S(z) = \bigl\{y \in Y\colon \psi(z, y) > \lambda\bigr\}, \quad z \in D.$$

By (5), $S(z)$ is open for each $z \in D$. We show that S is weakly G-KKM w.r.t. T. Suppose, on contrary, that there exists $A \in \langle D\rangle$ and $\bar{x} \in \Gamma(A)$ such that $T(\bar{x}) \cap S(A) = \emptyset$. Then for each $y \in T(\bar{x})$, $A \subset \{z \in D\colon \psi(z, y) \leqslant \lambda\}$. Hence, by (6),

$$\bar{x} \in \Gamma(A) \subset \bigl\{x \in X\colon \varphi(x, y) \leqslant \lambda\bigr\} \quad \text{for all } y \in T(\bar{x}).$$

Therefore, $\sup_{y \in T(\bar{x})} \varphi(\bar{x}, y) \leqslant \lambda$, which contradicts $\lambda < \beta$.

By Theorem 3, via Remark 2, for each $A \in \langle D \rangle$ there exists a point $x_A \in \Gamma(A)$ such that $T(x_A) \cap S(z) \neq \emptyset$ for all $z \in A$. Consequently, $\min_{z \in A} \sup_{y \in T(x_A)} \psi(z, y) > \lambda$, whence

$$\sup_{x \in \Gamma(A)} \min_{z \in A} \sup_{y \in T(x)} \psi(z, y) > \lambda \quad \text{for all } A \in \langle D \rangle,$$

which proves the first part of the theorem.

If $D \subset X$ and condition (4) is fulfilled then, for each $\lambda < \beta$ and $y \in T(x)$, the set $\{x \in X\colon \varphi(x, y) \leqslant \lambda\} = \bigcap_{\lambda < \alpha < \beta} \{x \in X\colon \varphi(x, y) < \alpha\}$ is Γ-convex, since an intersection of Γ-convex sets is Γ-convex, too. It is easily seen that (3) and (4) imply (6). $\square$

4. *G*-KKM property

A G-convex space $(Y; \Gamma)$ is called *C-space* (or *H-space*) if each $\Gamma(A)$ is assumed to be contractible or, more generally, infinitely connected (that is, n-connected for all $n \geqslant 0$) and if for each $A, B \in \langle Y \rangle$, $A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$. An *$LC$-metric space* $(Y; \Gamma, d)$ is a C-space $(Y; \Gamma)$ equipped with a metric d such that for any $\varepsilon > 0$, the set $B(C, \varepsilon) = \{y \in Y\colon d(y, C) < \varepsilon\}$ is Γ-convex whenever $C \subset Y$ is Γ-convex and all open balls are Γ-convex.

For examples and details on C-spaces see [8,9].

If X is a topological space and $Z \subset X$, $\dim_X Z \leqslant 0$ means that the covering dimension of F is $\leqslant 0$ for every set $F \subset Z$ which is closed in X (see [10]).

The following is due to Ben-El-Mechaiekh and Oudadess [4, Lemma 2].

Lemma 8. *Let X be a paracompact space, $(Y; \Gamma, d)$ an LC-metric space, $Z \subset X$ with $\dim_X Z \leqslant 0$, and $T : X \to 2^Y$ a l.s.c. mapping such that $T(x)$ is Γ-convex for all $x \notin Z$. Then for each $\varepsilon > 0$, T admits an ε-approximate selection, that is, a continuous function $g_\varepsilon : X \to Y$ such that $g_\varepsilon(x) \in B(T(x), \varepsilon)$ for all $x \in X$.*

Recall that if Y is a compact metric space and $\mathcal{F}$ is a finite closed cover of Y, then there exist a positive number, denoted by $\varepsilon(\mathcal{F}, Y)$ and called the *Lebesque number of $\mathcal{F}$* (see [1, p. 101]) having the following property:

for every nonempty set $Z \subset Y$ of diameter (diam Z) less than $\varepsilon(\mathcal{F}, Y)$, the set $\bigcap\{F \in \mathcal{F}\colon F \cap Z \neq \emptyset\}$ is nonempty.

Theorem 9. *Let $(X, D; \Gamma_1)$ be a paracompact G-convex space, $(Y; \Gamma_2, d)$ an LC-metric space, $Z \subset X$ with $\dim_Z X \leqslant 0$ and $T : X \to 2^Y$ a compact l.s.c. mapping with $T(x)$ Γ_2-convex for all $x \in X \setminus Z$. Then T has the G-KKM property.*

Proof. Let $S : D \to 2^Y$ be a generalized G-KKM mapping w.r.t. T. For simplicity we may suppose that S has closed values and then we have to prove that the family $\{S(z)\colon z \in D\}$ has the finite intersection property.

Let $A \in \langle D \rangle$. Since T is compact mapping, $Y_1 = \overline{T(\Gamma_1(A))}$ is a compact metric space. The mapping S being generalized G-KKM w.r.t. T we have $T(\Gamma_1(A)) \subset S(A)$, and since $S(A)$ is closed, $Y_1 \subset S(A)$.

Denote by $\mathcal{F}$ the closed cover of Y_1, $\{S(z)\cap Y_1\colon z\in A\}$ and choose a positive integer ε less than $\frac{1}{2}\varepsilon(\mathcal{F},Y_1)$. By Lemma 8, there exists a continuous function $g_\varepsilon: X\to Y$ such that $g_\varepsilon(x)\in B(T(x),\varepsilon)$. Define the mappings $\widetilde{T}:\Gamma_1(A)\to 2^Y$, $\widetilde{S}: D\to 2^Y$ by

$$\widetilde{T}(x)=\overline{B}\bigl(g_\varepsilon(x),\varepsilon\bigr),\qquad \widetilde{S}(z)=S(z)\cap Y_1.$$

One readily verifies that $\widetilde{T}$ is an u.s.c. mapping with closed values.

We have already seen, in the proof of Theorem 3, that if for each $B\in\langle A\rangle$ we put $\widetilde{\Gamma}_1(B)=\Gamma_1(A)\cap\Gamma_1(B)$, then $(\Gamma_1(A),A;\widetilde{\Gamma}_1)$ is a G-convex space. We prove that if we replace the G-convex space $(X,D;\Gamma_1)$ by $(\Gamma_1(A),A;\widetilde{\Gamma}_1)$ then $\widetilde{S}$ is weakly G-KKM w.r.t. $\widetilde{T}$. To this purpose let us consider $B\in\langle A\rangle$ and $x\in\widetilde{\Gamma}_1(B)$. On the one hand, since $T(x)\subset Y_1$ and S is G-KKM mapping w.r.t. T, we have $T(x)\subset S(B)\cap Y_1=\widetilde{S}(B)$. On the other hand, since g_ε is an ε-approximate selection of T, $T(x)\cap\widetilde{T}(x)\neq\emptyset$. Thus $\widetilde{T}(x)\cap\widetilde{S}(B)\neq\emptyset$, hence $\widetilde{S}$ is weakly G-KKM mapping w.r.t. $\widetilde{T}$.

By Theorem 3 and Remark 2, there exists a point $x_0\in\widetilde{\Gamma}_1(A)$ such that $\widetilde{T}(x_0)\cap(S(z)\cap Y_1)\neq\emptyset$ for each $z\in A$. Since $\operatorname{diam}\widetilde{T}(x_0)<\varepsilon(\mathcal{F},Y_1)$, it follows that $\bigcap\{S(z)\colon z\in A\}\neq\emptyset$. $\square$

Remark 3. From the previous proof, one can see that a compact mapping $T:(X,D;\Gamma)\to 2^Y$ (Y is a metric space) has the G-KKM property provided that for each $\varepsilon>0$ there is an u.s.c. closed-valued mapping $T_\varepsilon: X\to 2^Y$ such that $\operatorname{diam}T_\varepsilon(x)\leqslant\varepsilon$ and $T_\varepsilon(x)\cap T(x)\neq\emptyset$ for all $x\in X$.

If X and Y are topological spaces a function $\varphi: X\times Y\to\mathbb{R}$ is said to be *marginally l.s.c.* on X (see [3]) if for every open set $V\subset Y$, the function $x\to\sup_{y\in V}\varphi(x,y)$ is l.s.c. on X. It is clear that every function $\varphi: X\times Y\to\mathbb{R}$ l.s.c. on X is marginally l.s.c. on X; but the reverse implication is not true (see [3]).

Let X be a topological space, $(Y,D;\Gamma)$ a G-convex space with $D\subset Y$, and β a real number. We say that a function $\varphi: X\times Y\to\mathbb{R}$ is *almost G-β-quasiconcave* on Y if for each $\lambda<\beta$ there exists $Z_\lambda\subset X$ with $\dim_X Z\leqslant 0$ such that for any $x\in X\setminus Z_\lambda$ the set $\{y\in Y\colon \varphi(x,y)>\lambda\}$ is Γ-convex.

The last result is a Sion type minimax inequality.

Theorem 10. *Let $(X,D;\Gamma_1)$ be a paracompact G-convex space and $(Y;\Gamma_2,d)$ be a compact LC-metric space. Let $\psi: D\times Y\to\mathbb{R}$, $\varphi: X\times Y\to\mathbb{R}$ be two functions and $\beta=\inf_{x\in X}\sup_{y\in Y}\varphi(x,y)$. Suppose that*:

(i) *φ is marginally l.s.c. on X;*
(ii) *φ is almost G-β-quasiconcave on Y;*
(iii) *ψ is u.s.c. on Y;*
(iv) *for any $\lambda<\beta$, $y\in Y$ and each $A\in\langle\{z\in D\colon \psi(z,y)<\lambda\}\rangle$ one has $\Gamma(A)\subset\{x\in X\colon \varphi(x,y)\leqslant\lambda\}$.*

Then

$$\inf_{x\in X}\sup_{y\in Y}\varphi(x,y)\leqslant\sup_{y\in Y}\inf_{z\in D}\psi(z,y).$$

If $D \subset X$, the conclusion holds if condition (iv) *is replaced by the following conditions*:

(v) $\varphi(z, y) \leqslant \psi(z, y)$ *for all* $(z, y) \in D \times Y$;
(vi) φ *is* G-β-*quasiconvex on* X.

Proof. Let $\lambda < \beta$ be fixed and define $T : X \to 2^Y$, $S : D \to 2^Y$ by

$$T(x) = \{y \in Y\colon\ \varphi(x, y) > \lambda\}, \quad x \in X,$$
$$S(z) = \{y \in Y\colon\ \psi(z, y) \geqslant \lambda\}, \quad z \in D.$$

By (i), for each open subset V of Y the set

$$\{x \in X\colon\ T(x) \cap V \neq \emptyset\} = \Big\{x \in X\colon\ \sup_{y \in V} \varphi(x, y) > \lambda\Big\}$$

is open, hence T is l.s.c.

Condition (ii) implies that $T(x)$ is Γ_2-convex for each $x \in X \setminus Z_\lambda$. By (iii), S has closed values, and condition (iv) implies that S is a generalized G-KKM mapping w.r.t. T.

By Theorem 9, the family $\{S(z)\colon\ z \in D\}$ has the finite intersection property and, since Y is compact, there exists an $y_0 \in \bigcap_{z \in D} S(z)$. Consequently,

$$\sup_{y \in Y} \inf_{z \in D} \psi(z, y) \geqslant \inf_{z \in D} \psi(z, y_0) \geqslant \lambda,$$

and the proof is complete. □

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